Lyapunov exponents
from the 1960s to the 2020s

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A few recent books


Lyapunov stability

Consider the differential equation

$$x' = A(t)x + R(t, x), \quad R(t, 0) \equiv 0.$$  \hspace{1cm} (1)

The Lyapunov exponent function is defined by

$$\lambda(v) = \limsup_{t \to +\infty} \frac{1}{t} \log \|\Gamma(t)v\|,$$

where $t \mapsto \Gamma(t)$ is the fundamental solution of the linearized equation $x' = A(t)x$. 

Lyapunov stability theorem, 1892

$\lambda(v) < 0$ for every $v$ together with "Lyapunov regularity" implies that the constant solution $x(t) \equiv 0$ is exponentially stable for equation (1).
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\[ \lambda(v) < 0 \text{ for every } v \text{ together with “Lyapunov regularity” implies that the constant solution } x(t) \equiv 0 \text{ is exponentially stable for equation (1).} \]
Let $\nu$ be a probability measure on GL($d$), such that $g \mapsto \log \|g^{\pm 1}\|$ are in $L^1(\nu)$. Let $(g_n)_n$ be independent random variables in GL($d$), all with probability distribution $\nu$. There exist numbers $\lambda^- (\nu)$ and $\lambda^+ (\nu)$, called extremal Lyapunov exponents, such that

$$\lim_{n \to \infty} \frac{1}{n} \log \|g_n \cdots g_1\| = \lambda^+ (\nu) \quad \text{and} \quad \lim_{n \to \infty} -\frac{1}{n} \log \| (g_n \cdots g_1)^{-1} \| = \lambda^- (\nu),$$

$\nu$-almost surely. Moreover, $\lambda^- (\nu) \leq \lambda^+ (\nu)$. }

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Furstenberg–Kesten ergodic theorem, 1960

There exist numbers $\lambda_-(\nu)$ and $\lambda_+(\nu)$, called extremal Lyapunov exponents, such that

$$\lim_{n} \frac{1}{n} \log \|g_n \cdots g_1\| = \lambda_+(\nu)$$

and

$$\lim_{n} \frac{1}{n} \log \|(g_n \cdots g_1)^{-1}\| = \lambda_-(\nu)$$

$\nu$-almost surely. Moreover, $\lambda_-(\nu) \leq \lambda_+(\nu)$. 

Linear cocycles

Let \((M, \mu)\) be a probability space and \(f : M \to M\) be a measure-preserving map. A linear cocycle over \(f\) is a map \(F : \mathcal{V} \to \mathcal{V}\), where \(\pi : \mathcal{V} \to M\) is a finite dimension vector bundle, such that the diagram

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\begin{array}{ccc}
\mathcal{V} & \xrightarrow{F} & \mathcal{V} \\
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commutes, and whose action \(F_x : \mathcal{V}_x \to \mathcal{V}_{f(x)}\) on each fiber of \(\mathcal{V}\) is linear.
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Example

\(\mu = \nu^\mathbb{Z}\) a Bernoulli measure on \(M = \text{GL}(d)^\mathbb{Z}\), \(f : M \to M\) the left-translation (shift), \(\mathcal{V} = M \times \mathbb{R}^d\) a trivial vector bundle, and

\[F(g, \nu) = (f(g), g_0 \nu),\quad \text{where } g = (g_n)_n.\]
Assume that the functions $x \mapsto \log \| F_x^{\pm 1} \|$ are in $L^1(\mu)$ and $f : M \to M$ is invertible.

**Oseledets multiplicative ergodic theorem, 1968**

For $\mu$-almost every $x \in M$, there are numbers $\lambda_1(x) > \cdots > \lambda_k(x)$, called Lyapunov exponents, and a splitting $\mathcal{V}_x = E^1_x \oplus \cdots \oplus E^k_x$ such that

$$\lim_{n \to \pm \infty} \frac{1}{n} \log \| F^n_x v \| = \lambda_j(x) \text{ for } v \in E^j_x.$$ 

Moreover, $F_x(E^j_x) = E^j_{f(x)}$ and $\lambda_j(x) = \lambda_j(f(x))$ at $\nu$-almost every point.
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For $\mu$-almost every $x \in M$, there are numbers $\lambda_1(x) > \cdots > \lambda_k(x)$, called **Lyapunov exponents**, and a splitting $\mathcal{V}_x = E_{x}^{1} \oplus \cdots \oplus E_{x}^{k}$ such that

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\lim_{n \to \pm \infty} \frac{1}{n} \log \|F_x^n v\| = \lambda_j(x) \text{ for } v \in E_{x}^{j}.
$$

Moreover, $F_x(E_x^j) = E_{f(x)}^j$ and $\lambda_j(x) = \lambda_j(f(x))$ at $\nu$-almost every point.

The heart of the proof is showing that $\mu$-almost every orbit has Lyapunov regularity.
Let $f : M \rightarrow M$ be a diffeomorphism and $F = Df : TM \rightarrow TM$ be the derivative.

We call $(f, \mu)$ **non-uniformly hyperbolic** if the Lyapunov exponents of $F = Df$ are non-zero $\nu$-almost everywhere.

Then there is a (measurable) hyperbolic dichotomy $T_x M = E_x^s \oplus E_x^u$, where

$$E_x^s = \bigoplus_{\lambda_j(x) < 0} E_x^j$$

and

$$E_x^u = \bigoplus_{\lambda_j(x) > 0} E_x^j.$$
Pesin stable manifold theorem, 1976

There is a measurable family of embedded smooth disks $W^{s}_{loc}(x)$ tangent to $E_{x}^{s}$ at $\nu$-almost every point and consisting of points that are forward-asymptotic to $x$.

Applying the theorem to the inverse $f^{-1}$, we get a corresponding statement for $E_{x}^{u}$.
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This is the starting point of one of the most fruitful areas of smooth dynamics, called Pesin (or non-uniform hyperbolicity) theory.

**Question:** How general is non-uniform hyperbolicity (almost everywhere non-vanishing Lyapunov exponents) among dynamical systems?
A diffeomorphism $f : M \to M$ is partially hyperbolic if there exists a continuous decomposition

$$T_x M = E^u_x \oplus E^c_x \oplus E^s_x$$

(defined at every point) which is invariant under the dynamics:

$$Df_x(E^*_x) = E^*_{f(x)} \text{ for all } * \in \{u, c, s\}$$

and ...
Partially hyperbolic dynamics

- $E^s$ is uniformly contracting:

$$\|Df_x \mid_{E^s_x}\| \leq \lambda < 1$$

- $E^u$ is uniformly expanding:

$$\|(Df_x \mid_{E^u_x})^{-1}\| \leq \lambda < 1$$

- $E^c$ is “in between”:

$$\frac{1}{\lambda} \frac{\|Df_x(v^s)\|}{\|v^s\|} \leq \frac{\|Df_x(v^c)\|}{\|v^c\|} \leq \lambda \frac{\|Df_x(v^u)\|}{\|v^u\|}$$
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**Question:** How often are the center (i.e. along $E^c$) Lyapunov exponents non-vanishing?
Examples of partial hyperbolicity

Fact: Partial hyperbolicity is an open property.
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- Take $A \in \text{SL}(d, \mathbb{Z})$ whose spectrum intersects the interior, the boundary, and the exterior of the unit disk in $\mathbb{C}$. Then the *induced map* is partially hyperbolic:

  $$ f_A : \mathbb{T}^d \to \mathbb{T}^d, \quad \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d $$
Examples of partial hyperbolicity

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- Let $f^t : M \to M, \ t \in \mathbb{R}$ be an **Anosov flow**: there is an invariant decomposition

$$T_x M = E^u_x \oplus \mathbb{R}X(x) \oplus E^s_x, \quad X = \text{associated vector field.}$$

Then the **time–1 map** $f^1$ is partially hyperbolic.
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  Then the time–1 map $f^1$ is partially hyperbolic.

- Let $g : N \to N$ be Anosov. Then any isometry extension is partially hyperbolic:

  $$f : N \times \mathbb{T}^d \to N \times \mathbb{T}^d, \quad f(x, v) = (g(x), v + \omega(x)).$$
Smooth cocycles

Let \((M, \mu)\) be a probability space and \(f : M \rightarrow M\) be a measure-preserving map.

A smooth cocycle over \(f\) is a map \(\widetilde{\mathcal{F}} : \mathcal{E} \rightarrow \mathcal{E}\), where \(\pi : \mathcal{E} \rightarrow M\) is fiber bundle whose fibers are Riemannian manifolds, such that the diagram

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\mathcal{E} & \xrightarrow{\widetilde{\mathcal{F}}} & \mathcal{E} \\
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commutes, and whose action \(\widetilde{\mathcal{F}}_x : \mathcal{E}_x \rightarrow \mathcal{E}_{f(x)}\) on each fiber of \(\mathcal{E}\) is a diffeomorphism.
Let $(M, \mu)$ be a probability space and $f : M \to M$ be a measure-preserving map.

A smooth cocycle over $f$ is a map $\tilde{F} : E \to E$, where $\pi : E \to M$ is fiber bundle whose fibers are Riemannian manifolds, such that the diagram

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$$

commutes, and whose action $\tilde{F}_x : E_x \to E_{f(x)}$ on each fiber of $E$ is a diffeomorphism.

**Example**

The projectivization $\tilde{F} : E \to E$ of a linear cocycle $F : V \to V$: the fibers $E_x = \mathbb{P}(V_x)$ and each $\tilde{F}_x$ is the projectivization of the linear map $F_x$. 
For \( z \in \mathcal{E} \) and \( v \) a tangent vector to the fiber at \( z \), the extremal Lyapunov exponents are

\[
\lambda_+(z, v) = \lim_{n \to \infty} \frac{1}{n} \log \| D\tilde{F}_z^n(v) \|.
\]

\[
\lambda_-(z, v) = \lim_{n \to \infty} -\frac{1}{n} \log \| D\tilde{F}_z^n(v)^{-1} \|.
\]

The limits exist \( m \)-almost everywhere if \( m \) is an \( \tilde{F} \)-invariant probability measure (Kingman subadditive ergodic theorem, 1968).

We are only interested in measures \( m \) that project down to \( \mu \) under \( \pi : \mathcal{E} \to M \).
Avila–Viana invariance principle, 2010

Let $\mathcal{A}$ be a generating $\sigma$-algebra in $M$ such that both $f$ and $x \mapsto \mathcal{F}_x$ are $\mathcal{A}$-measurable. If $\lambda_-(z, v) \geq 0$ at $m$-almost every point then the disintegration $x \mapsto m_x$ of $m$ along the fibers is $\mathcal{A}$-measurable.

Applying the theorem to the inverse, we get a dual statement when $\lambda_+(\mathcal{F}, z, v) \leq 0$.

This extends results of Furstenberg, Ledrappier and Bonatti–Viana for linear cocycles.
Let us go back to the partially hyperbolic setting:

\[ f : M \rightarrow M \] with invariant splitting \( TM = E^s \oplus E^c \oplus E^u \).
Let us go back to the partially hyperbolic setting:

\[ f : M \to M \text{ with invariant splitting } TM = E^s \oplus E^c \oplus E^u. \]

The center Lyapunov exponents of \( f \) are the numbers

\[ \lambda(v^c) = \lim_{n \to \infty} \frac{1}{n} \log \| Df^n_x(v^c) \| \text{ of vectors } v^c \in E_x^c \]

They are well defined almost everywhere (Oseledets theorem).
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They are well defined almost everywhere (Oseledets theorem).

Question: Can we always perturb \( f \) to make the center Lyapunov exponents non-zero?
Let $f_A : \mathbb{T}^4 \to \mathbb{T}^4$ be induced by some linear map $A \in \text{SL}(4, \mathbb{Z})$ with exactly two eigenvalues in the unit circle.

Basic facts:

- $f_A$ preserves some (constant) symplectic form $\omega$.
- $f_A$ preserves volume.
- Assuming that no eigenvalue is a root of unit, $f_A$ is ergodic.
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**F. Rodriguez-Hertz stable ergodicity theorem, 2005**

Every volume-preserving diffeomorphism $f$ close to $f_A$ is ergodic.
Symplectic diffeomorphisms

In fact, $f$ is **stably Bernoulli** among symplectic diffeomorphisms:

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**Avila, Viana stable Bernoulli theorem, 2010**

Let $f : \mathbb{T}^4 \to \mathbb{T}^4$ be any $\omega$-symplectic diffeomorphism close to $f_A$. Then:

- either $f$ has all center Lyapunov exponents non-zero,
- or $f$ is conjugate to $f_A$ by a volume-preserving diffeomorphism.

In either case, $f$ is ergodically equivalent to a Bernoulli shift.
In fact, $f$ is stably Bernoulli among symplectic diffeomorphisms:

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In either case, $f$ is ergodically equivalent to a Bernoulli shift.

The proof involves several applications of the invariance principle.
A different application of the invariance principle yields a direct proof that vanishing exponents can be disposed of, at least in some cases:

Let $f : M \to M$ be a partially hyperbolic, symplectic, $C^k$ diffeomorphism having some periodic point.

**Marín, 2016**

Assume that $f$ is accessible, center-bunched and pinched and the center bundle $E^c$ is 2-dimensional. Then $f$ is $C^k$-approximated by non-uniformly hyperbolic symplectic diffeomorphisms.
Many other important issues

- simplicity of the Lyapunov spectrum
- dependence of the Lyapunov exponents on the cocycle
- Schrödinger cocycles, random or quasi-periodic
- dynamics of group actions
- numerical analysis of Lyapunov exponents
- ...
Lyapunov exponents